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SOME EXPERIMENTS ON THE STABILITY OF CIRCULAR, RUBBER BARS UNDER END THRUST

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Abstract—Some experiments concerning stability of thick, circular, rubber bars subject to compressive end loads are described, and the data is compared with various analytical solutions that are based upon general theories of elastic stability. The experiments indicate existence of a critical transition slenderness ratio that separates the familiar Euler mode of buckling from certain forms of radially symmetric instability. Theoretical estimates obtained by Beatty [1] are found to be in remarkably good qualitative agreement with the data. It is demonstrated that agreement with analyses of Fosdick and Shield [2], Read [3] and the well-known classical formulae for Euler collapse also is quite good, even for relatively short bars—but only for critical compression ratios greater than the transition slenderness ratio. For radially symmetric buckling, it is shown that the result of Wilkes [4] fails of agreement with experimental observations.

1. INTRODUCTION

THIS paper is concerned with an experimental investigation of stability of flat end, circular bars of various slenderness ratios under uniform, compressive end loads. As is well-known, such experiments are usually limited to thin bodies of classical Hookean materials that fail by buckling in bending at a compression stress that does not exceed the proportional limit of the material; the purpose being to show that under these circumstances the experimental values fall upon the so-called Euler curves. This is not our objective.

Our purpose is to study the deformation behaviour of thick bars of extruded and molded natural rubbers subject to (possibly) finite compression in light of recent theoretical results for neo-Hookean bodies obtained by Beatty [1], Fosdick and Shield [2], Read [3] and Wilkes [4]. All of these investigations are based upon general theories of elastic stability, and contain as special cases the familiar classical results or estimates of them. Although experimental data on the deformation and rupture* of rubber-like bodies of various geometries and subject to diverse loadings is abundant (e.g. [5–10]), we know of no previous experimental studies of the kind reported here.

Further motivation for our experiments is presented in Section 2 where we sketch the theoretical results gotten by Beatty [1], examine their implications, and discuss their physical interpretation. The materials, apparatus and experimental procedure are described in Section 3. In Section 4 the specific experiments are discussed and the relevant

* Gent and Lindley [6], for example, have investigated an unusual rupture process that occurs during extension of thin rubber wafers bonded to metal end pieces. They report on a material instability that corresponds to the sudden formation of internal cracks at a certain well-defined and relatively small tensile load.

data is tabulated. We show here also that for the range of deformation ratios encountered in the experiments the ideal neo-Hookean model serves as an approximate but not entirely satisfactory analytical description of the materials employed. The theoretical results obtained by Fosdick and Shield [2], Read [3] and Wilkes [4] are outlined in Section 5, wherein we also compare our experimental curves with theoretical values gotten from [1–4]. The classical Euler results are also mentioned here.

2. CRITICAL LOAD AND DEFORMATION FORMULAE FOR CIRCULAR COLUMNS

We begin with a brief review of certain results obtained by Beatty [1] from a general stability theory based upon an energy criterion. Beatty has found *estimates* for the critical deformation ratio λ_{cr} and the critical load P_{cr} for neo-Hookean columns of *arbitrary*, uniform cross-section and length when the stress that results from dead loads is uniform throughout the body. In particular, for compression of a circular cylinder of radius R and length L in the deformed critical state, he finds

$$\left.\begin{aligned} \lambda_{cr} &= (1+\mu)^{-\frac{1}{3}} \\ P_{cr} &= AE\mu/3(1+\mu)^{\frac{2}{3}} \end{aligned}\right\} \quad \text{for } \mu \leq 1, \tag{1}$$

and

$$\begin{aligned} \lambda_{cr} &= [2\mu/(3\mu+1)]^{\frac{1}{3}} \\ P_{cr} &= AE[(\mu+1)/6\mu]\,[2\mu/(3\mu+1)]^{\frac{2}{3}} \end{aligned} \quad \text{for } \mu \geq 1, \tag{2}$$

where $\mu = 3R^2/L^2$, $A = \pi R^2$ and E is the elastic modulus. To relate R and L to the corresponding quantities R_0 and L_0 in the natural, undeformed state, we use the relations

$$L = \lambda_{cr} L_0, \quad R = \lambda_{cr}^{-\frac{1}{2}} R_0 \tag{3}$$

in equations (1) and (2), and get

$$\left.\begin{aligned} \lambda_{cr} &= (1-\mu_0)^{\frac{1}{3}} \\ P_{cr} &= (A_0 E\mu_0/3)(1-\mu_0)^{-\frac{2}{3}} \end{aligned}\right\} \quad \text{for } 2\mu_0 \leq 1, \tag{4}$$

and

$$\left.\begin{aligned} \lambda_{cr} &= [\mu_0(\alpha_0-3)/2]^{\frac{1}{3}} \\ P_{cr} &= A_0 E(\alpha_0-1)[(\mu_0/2)(\alpha_0-3)]^{\frac{1}{3}}/12 \end{aligned}\right\} \quad \text{for } 2\mu_0 \geq 1, \tag{5}$$

where

$$\mu_0 = 3R_0^2/L_0^2, \quad A_0 = \pi R_0^2 \quad \text{and} \quad \alpha_0 = (9+8/\mu_0)^{\frac{1}{2}}.$$

We note that the limiting case $\mu = 1$ (or $2\mu_0 = 1$) corresponds to a deformed geometry for which all principal values of the inertia tensor $I_{ij} = \int x_i x_j \,\mathrm{d}V$ are equal. In this case we find

$$\lambda_{cr} = 3P_{cr}/A_0 E = 2^{-\frac{1}{3}}. \tag{6}$$

It is easy to see from equations $(1)_1$ and $(2)_1$ that λ_{cr} decreases as the slenderness ratio L/R decreases to the value $\sqrt{3}$ and afterwards increases as L/R approaches zero; the minimum value of λ_{cr} being given by (6) at $L/R = \sqrt{3}$, or $L_0/R_0 = \sqrt{6}$. Thus, Beatty's theory estimates that all neo-Hookean circular columns under uniform axial dead loads are unstable for compression ratios $\lambda \leq 0{\cdot}793$. Furthermore, it can be shown that as $L/R \to 0$, $\lambda_{cr} \to (\frac{2}{3})^{\frac{1}{3}}$ and $d\lambda_{cr}/d(L/R) \to 0$, which corresponds to a relative maximum for λ_{cr}. It thus follows that all neo-Hookean cylindrical struts with $L_0/R_0 < \sqrt{6}$ will be stable for compression ratios $\lambda > 0{\cdot}874$.

It was assumed in the derivation of equations (1) and (2) that the deformation is homogeneous for all end loads not greater than P_{cr}. Consequently, it is reasonable to expect that the foregoing theoretical results are estimates of the critical values of λ and P for geometrical or structural instability as opposed to material instability. Now it is certainly well established that a sufficiently long strut will fail of stability by bending, whereas a very short bar might exhibit some kind of bulging effect. Thus, an obvious but strictly intuitive interpretation of the theory is that it yields as estimate a limiting slenderness ratio $L_0/R_0 = \sqrt{6}$ that separates instability by bending from instability by swelling. Moreover, the transition is marked by a homogeneously deformed state in which all principal values of inertia are equal. Nevertheless, the precise physical nature of the instability within the various ranges indicated is not clear, because the kinds of geometry alterations that may be expected are not determined. On the other hand, nowhere have we been able to find an analysis based upon a general theory of elastic stability that does predict changes in geometry that result at the onset of instability. Of course, we expect to find in a compression test some kind of bending or bulging effect; but theoretical conditions that tell us when we shall have one or the other of these cases have apparently never been discovered. It is surely wrong to suppose that only a very long, slender column will exhibit Euler buckling when the load is sufficiently large, for example. It is plain that the foregoing theoretical results are not restricted in this way.

The experiments described in the remainder of this paper support the conclusion that the inequalities that separate equations (1) and (2) also separate instability due to buckling in bending from instability due to buckling in a radially symmetric mode. We again emphasize that Beatty's theory gives only estimates of critical values. In particular, we note that when the usual slenderness assumption is introduced, the theory yields as estimate a critical load roughly equal to forty percent of the classical Euler load [1, 3]. Consequently, we can expect, at best, only qualitative agreement with observed overall material behavior.

3. EXPERIMENTAL APPARATUS AND PROCEDURE

Simple compression experiments on flat end columns made of natural rubber were performed in the usual way by placing a specimen between lubricated platens. Circular columns of various slenderness ratios were investigated. The natural dimensions of all samples are given in Table 1, where the letter prefix in the sample designation indicates the material from which the specimen was prepared. The letters M, M* denote two kinds of molded, carbon-filled natural rubber, and E designates an extruded, unfilled rubber. Unfortunately, the exact composition of the materials is not known.

A prime consideration in the preparation of the test-pieces was the necessity of producing straight bars with flat ends, mutually parallel and perpendicular to the major axis

TABLE 1. SUMMARY OF EXPERIMENTAL DATA FOR CIRCULAR COLUMNS UNDER END THRUST
EULER BUCKLING $L_0/R_0 \geq 4{\cdot}62$; SYMMETRIC BUCKLING $L_0/R_0 \leq 4{\cdot}39$

Sample designation	Natural geometry (in.) L_0	R_0	L_0/R_0	E (lb/in^2)	λ_{cr}	P_{cr} (lb)	Photo time-lapse (sec)	P_{max} (lb)
M-1	4·922	0·507	9·70	638	0·961–0·955	21·1– 25·2	12	55
M-2	3·545	0·510	6·95	638	0·898–0·888	53·8– 60·0	12	118
M-6	4·100	0·516	7·95	638	0·933–0·925	36·1– 40·5	10	84·5
M-7	2·986	0·517	5·78	638	0·845–0·816	98·5–107·0	12	184·5
M*-1	6·960	0·663	10·50	563	0·965–0·958	29·8– 34·8	6	75
M*-2	3·088	0·658	4·69	563	0·818–0·800	171·0–193·0	20	520
M*-3	3·079	0·666	4·62	563	0·790–0·772	202·0–223·0	18	567
E-1	2·655	0·406	6·54	141	0·914–0·900	7·2– 8·5	12	19·3
E-2	1·968	0·408	4·82	141	0·750–0·739	24·1– 25·4	6	39·2
M-3	1·950	0·514	3·79	638	0·919–0·907	44·0– 49·3	6	
M-4	1·291	0·509	2·54	638	0·944–0·925	34·5– 43·5	6	
M-5	2·220	0·505	4·39	638	0·770–0·758	141·0–150·0	9	
M-8	2·163	0·505	4·28	638	0·841–0·815	88·0–101·5	15	
M-9 (fixed ends)	4·100	0·516	7·95	638	0·934–0·928	36·6– 41·2	10	82

of the column.* We were able to manufacture excellent specimens by machine grinding. Bars prepared in this way, when placed on end upon a horizontal surface, exhibited a maximum deviation from the true vertical of less than 0·002 in. from end to end. With the aid of machine tools, precise circumferential and axial scaling lines were ruled on the samples. We also note that parallelism to within 0·001 in. over the entire surfaces of the anvil and load cell of the test machine was obtained by meticulous arrangement of the apparatus. We had no other control over possible eccentricity in the end loads. However, because the ends of the bars would not be free to rotate during compression, it was our opinion that a minute eccentricity in the applied loads ought not to alter significantly the experimentally determined critical values, except possibly in the neighborhood of the transition region, if one exists.

The experimental arrangement employed for the compression tests is shown schematically in Fig. 1. The test-piece was placed between the movable crosshead and fixed compression load cell of the machine and loaded at a constant rate of 0·2 in/min. The compressive load was recorded continuously as a function of time on a chart recorder that was synchronized to start precisely when the crosshead motion started. Because we were unable to predict in advance the plane of bending, a mirror was positioned adjacent to the column so that two axial grid lines, 90° apart, could be observed simultaneously. Friction between the column ends and the machine anvils was reduced by lubricating these surfaces. The subsequent deformation of the samples was photographed sequentially.

The experimental procedure was as follows. With the sample under a very small compressive load a photograph was taken to record the initial sample configuration.

* We note that the stock from which the extruded rods were fabricated exhibited a pronounced axial curvature which probably resulted from the curing process after extrusion. However, we were able to eliminate this defect by heating a suitable length at 120°C for several hours while maintaining the rubber under slight tension.

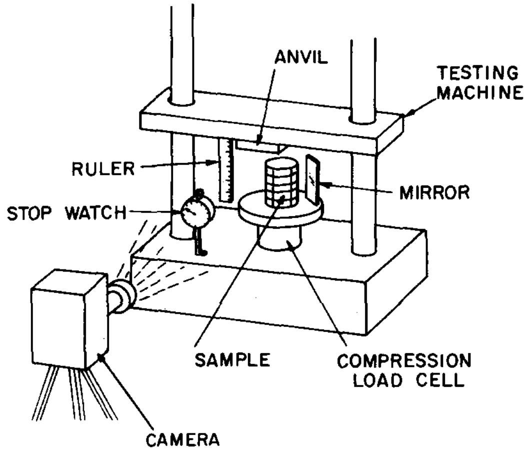


FIG. 1. Experimental arrangement.

The stopwatch and the crosshead were then started, and a sequence of photographs was taken manually as the test proceeded. Each photograph showed the sample, its mirror image, the stopwatch showing the elapsed time, and a ruler. The clock permitted time synchronization of the photographs with the force–time record, and the scale provided means for accurate determination of the deformation of the specimen. The photographic negatives were analyzed by projecting the pictures onto an optical comparator. This resulted in a magnification of from 4:1 to 5:1; that is, 1·0 in. on the test-piece projected as 4·0–5·0 in. on the comparator. By this method, the dimensions of a strut in any configuration could be determined with an accuracy of $\pm 0{\cdot}004$ in. We emphasize that this value is determined only by the scale and the magnification factor employed; it represents our maximum error in reading the scale. This deviation being the same for all samples, regardless of their size, our error in determining the deformations will be greatest for the shorter bars. Interpreted in terms of the compression ratio, we may expect a maximum error of 1% in the measured values of λ. This is a conservative value in the sense that it conforms roughly to the worst conditions experienced in all tests.

4. EXPERIMENTAL RESULTS

Since the theoretical stability analyses in [1–4] apply only to bodies in a state of uniform strain prior to instability, the experimental critical values of λ and P have been determined on the following basis. *Any discernible steadily increasing deviation from a state of homogeneous strain in a major central portion* of a specimen under simple compressive end loads constitutes a primary instability; and the ultimate state of homogeneous strain defines the critical state of deformation.* Of course, the experiments were not limited to a search for these values alone; all photographs obtained in the course of this investigation were thoroughly analyzed so as not to exclude the possibility of our observing other kinds of behavior. The pertinent experimental data gotten from the pictures is given in Table 1 and will be explained subsequently.

* The major central portion of a test-piece constituted about $\frac{2}{3}$ its total length in any configuration.

4.1 *Instability due to buckling by bending*

A typical series of photographs that illustrate various states of deformation that occurred during the compression of cylindrical columns that exhibited Euler buckling is shown in Fig. 2. The corresponding compressive load vs. time behavior is illustrated in Fig. 3, where the lettered points on the curve conform with the states of deformation

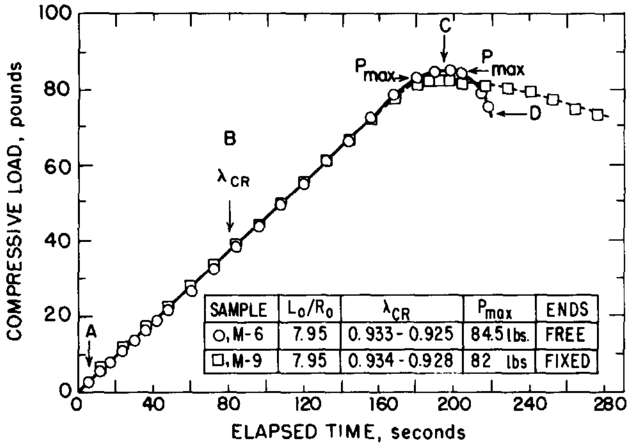
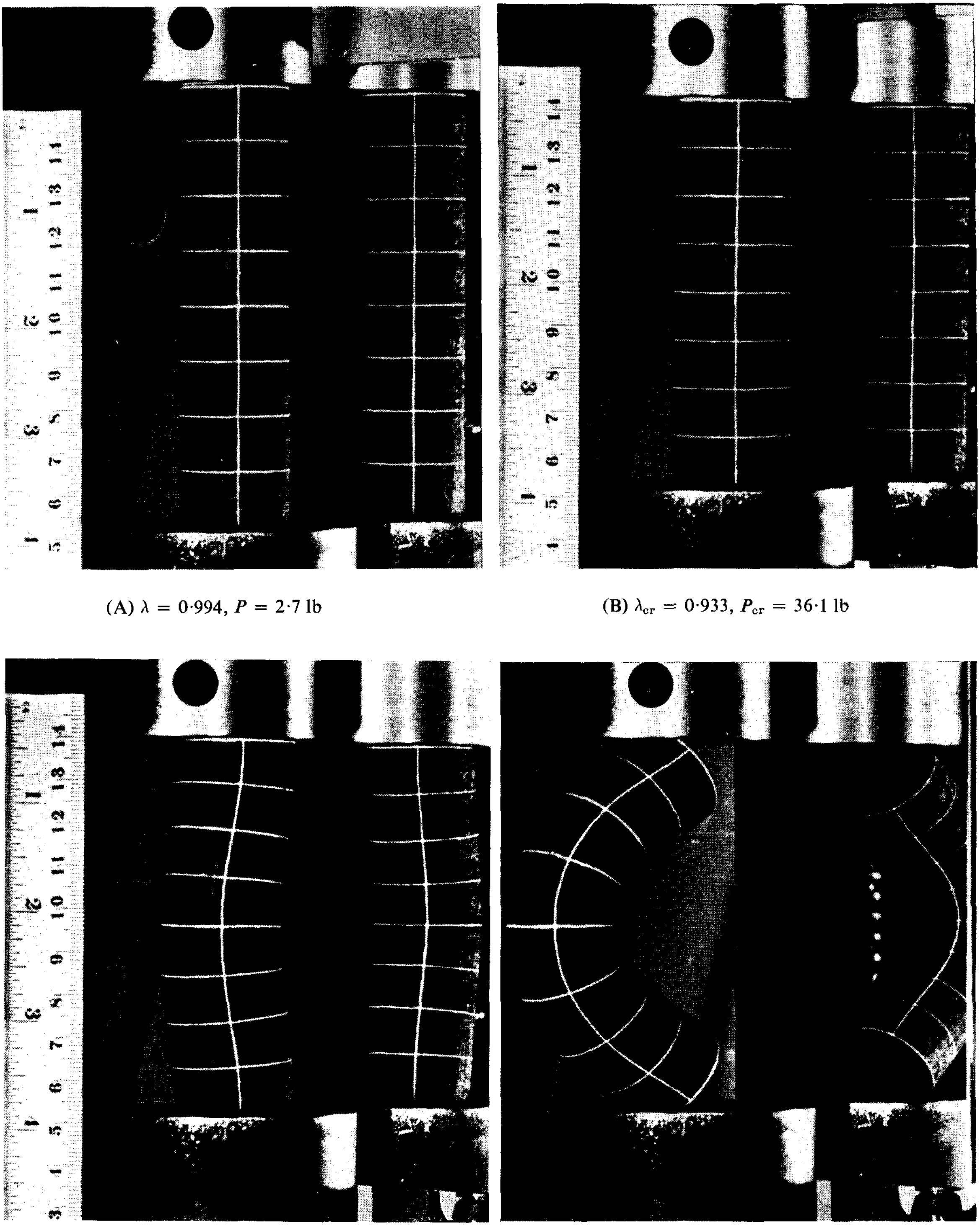


FIG. 3. Typical compressive load–time data for rubber circular columns exhibiting Euler buckling.

shown in the photographs. In accordance with our criterion, the critical values of λ and P correspond to the onset of curvature or bending of the axial grid marks. Figure 2(B) portrays a typical primary critical state from which a definite steadily increasing curvature could be observed. Of course, the larger magnification employed in the data reduction permitted a more definite evaluation of the critical state than is possible from these relatively small illustrations. Notice that the onset of instability results in no discernible inflection or discontinuity in the force–time curve. As opposed to the usual catastrophic failure of steel columns, the onset of instability in rubber columns is a very subtle effect, and accurate visual verification of it is generally quite difficult. In addition, the time-lapse between sequential photographs introduces a small error in ascertainment of the ultimate homogeneous state of deformation; consequently, a range of values for λ_{cr} and P_{cr} is given in the data listed in Table 1. This range was gotten from two sequential photographs that showed the test-piece first in an apparent state of homogeneous strain followed by a non-uniform state in which the axial grid line was definitely curved and continued to bend under increasing load. In spite of lubrication, the material consistently tended to adhere to the end plates of the test machine, but away from the ends, in the major central portion of the specimen, there was no apparent departure from a homogeneous deformation other than that described here.

After commencement of primary instability, the loading was continually increased and subsequently attained a well-defined maximum value, P_{max}, as shown in Fig. 3. Thereafter, the column continued to bend with a decreasing load until it abruptly snapped into a new configuration, as illustrated in Fig. 2(D). This collapse behavior was characteristic of all bars that exhibited Euler buckling.



(A) $\lambda = 0{\cdot}994$, $P = 2{\cdot}7$ lb

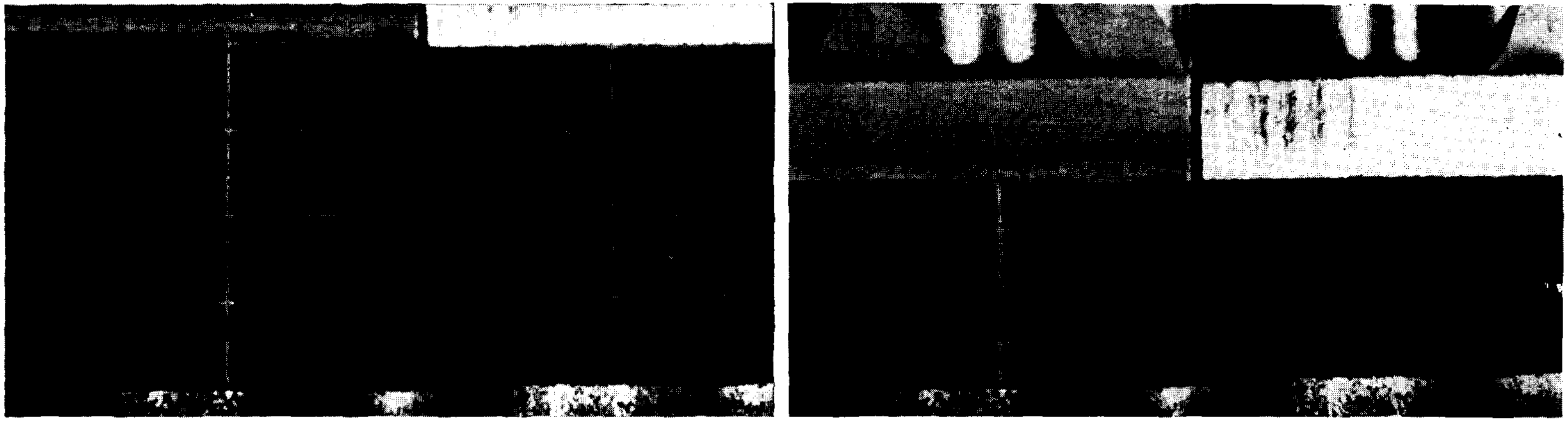
(B) $\lambda_{cr} = 0{\cdot}933$, $P_{cr} = 36{\cdot}1$ lb

(C) $P_{max} = 84{\cdot}5$ lb

(D) After P_{max}

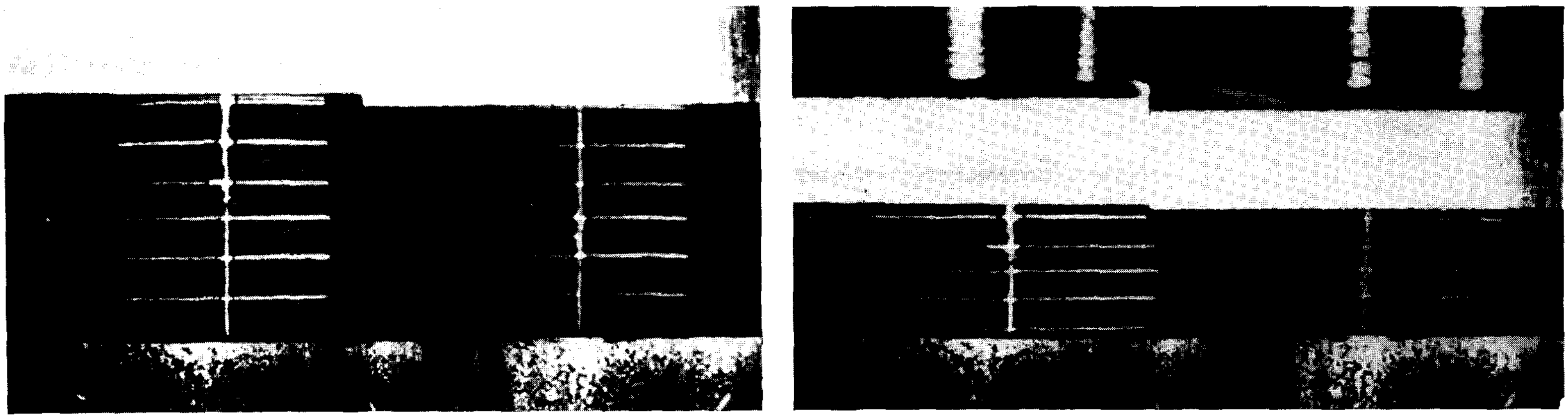
FIG. 2. Cylindrical column M-6, $L_0/R_0 = 7{\cdot}95$.

[*facing p.* 628



(A) $\lambda_{cr} = 0{\cdot}907$, $P_{cr} = 49{\cdot}3$ lb (B) $P = 475$ lb

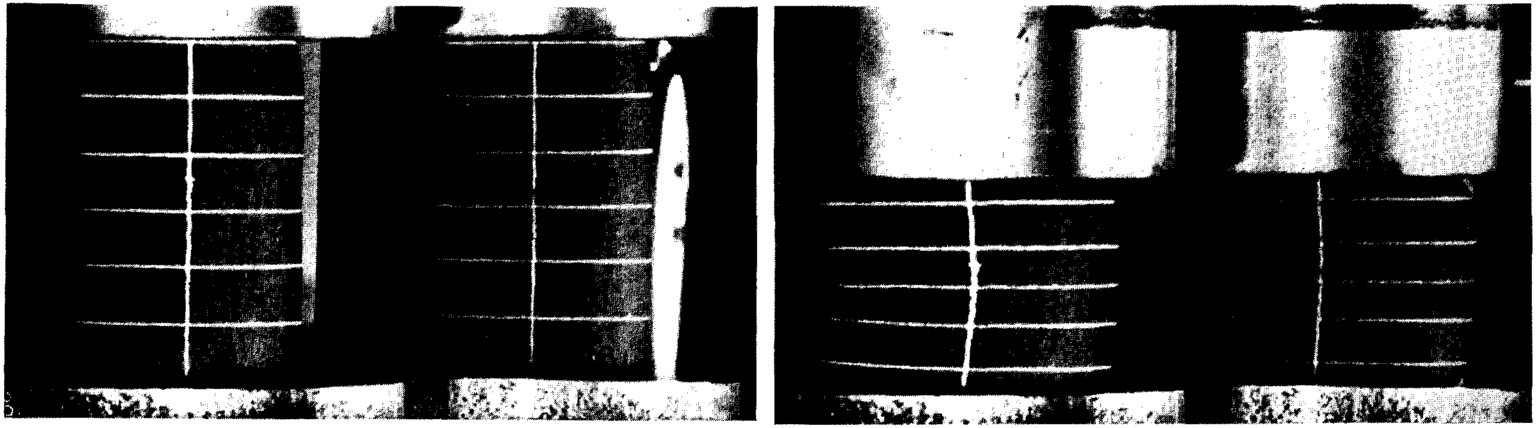
Specimen M-3, $L_0/R_0 = 3{\cdot}79$



(C) $\lambda_{cr} = 0{\cdot}925$, $P_{cr} = 43{\cdot}5$ lb (D) $P = 515$ lb

Specimen M-4, $L_0/R_0 = 2{\cdot}54$

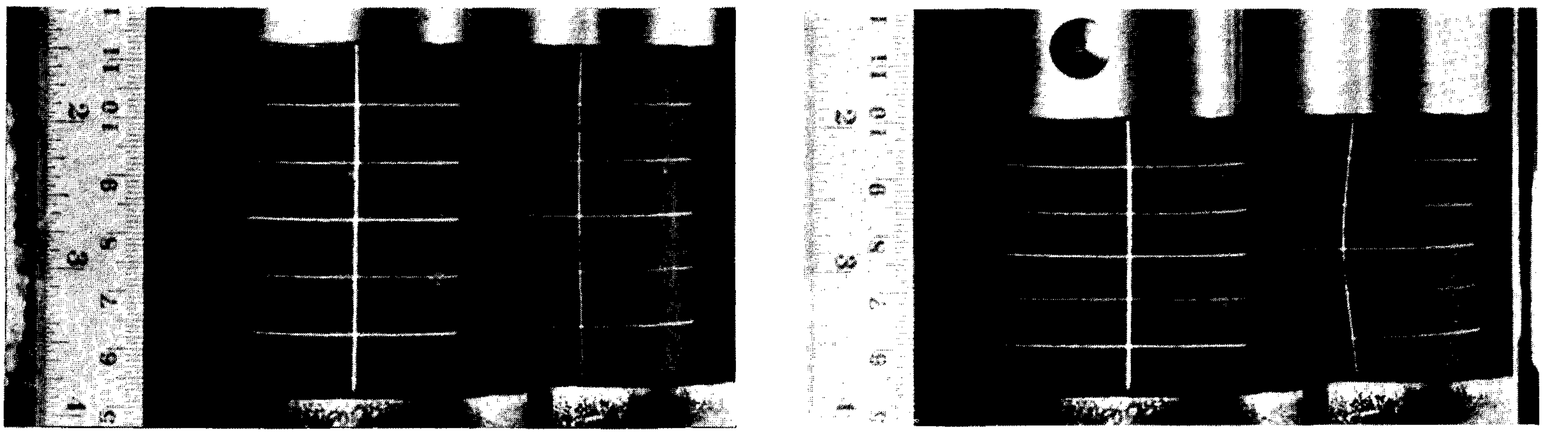
FIG. 4. Radially symmetric buckling of short circular cylinders.



(A) $\lambda_{cr} = 0{\cdot}758$, $P_{cr} = 150$ lb (B) $P = 562$ lb

Specimen M-5, $L_0/R_0 = 4{\cdot}39$

FIG. 6. Radially symmetric buckling in region of transition.



(A) $\lambda_{cr} = 0{\cdot}772$, $P_{cr} = 223$ lb (B) $P = 393$ lb

Specimen M*-3, $L_0/R_0 = 4{\cdot}62$

FIG. 7. Euler buckling in region of transition.

Since the columns were not free to rotate at the ends and, though well lubricated, tended to adhere to the machine platens, when the critical load was exceeded, the resultant end loads produced a buckling mode similar to that of the classical fixed end column. In view of these conditions an experiment was performed on a sample identical to M-6 but having its ends bonded to metal plates that were rigidly fastened in the machine. The compressive deformation behavior of this test-piece, designated M-9, was similar to that shown in Fig. 2; and the load–time data as compared with the identical free end bar is shown in Fig. 3. As might be expected, the presence of flat ends caused the free end column to behave overall as a fixed end strut. Indeed, both columns exhibited essentially the same critical values and the same maximum load. Since bonded ends prevented the specimen M-9 from snapping into another configuration, bending deformation continued beyond the maximum load but required a steadily decreasing thrust until eventually the ends pulled loose from the platens.

4.2 *Instability due to radially symmetric buckling*

A similar sequence of photographs that demonstrate the compressive deformation typical of shorter cylinders is shown in Fig. 4. As indicated, such columns did not buckle in the Euler sense but exhibited a bulging effect, as is intuitively natural. Notice that the specimen M-3, which had a slenderness ratio slightly greater than that of M-4, behaved differently. Although initially the specimen started to swell uniformly at the mid-section, similar to the behavior of M-4, an apparent radially symmetric necking or rippling of the lateral surface developed upon further deformation. The sample subsequently exhibited a minute axial curvature but remained necked at the mid-plane. Such primary behavior seems to correspond to instability of the type investigated by Wilkes [4].

The corresponding compressive load–time data for these samples is shown in Fig. 5.

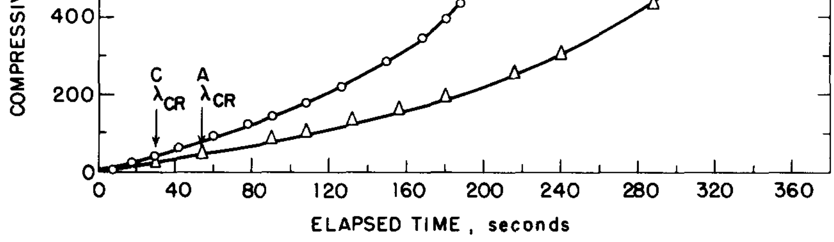


FIG. 5. Typical compressive load–time data for rubber, circular columns exhibiting radially symmetric buckling.

It is apparent that these curves are quite different from those found for the longer columns. In the present situation, the maximum impressed load was limited by the capacity of the testing apparatus (1000 lb) and not by any apparent material collapse. In accordance with our experimental stability criterion, the critical state in this case was marked by incipient bulging of the lateral surfaces of the test-piece; however, for such short bars the critical state was extremely difficult to determine by our present technique. We are unable to assess in any case the extent to which end effects may have influenced our measurements.

4.3 *Summary of experimental results*

We have found from these particular experiments that cylindrical rods having a slenderness ratio $L_0/R_0 \geq 4{\cdot}62$ fail of stability in the Euler sense. Moreover, it is important to note that all such bars sustain a characteristic maximum end thrust prior to ultimate collapse. Of course the foregoing equality represents only the minimum value found for the columns tested and does not necessarily constitute the minimum slenderness ratio for which Euler buckling will occur. On the other hand, we have also found that cylindrical rods with $L_0/R_0 \leq 4{\cdot}39$ do not initially exhibit Euler buckling. Consequently, what we have established for the columns tested are bounds for an experimental estimate of what appears to be a limiting slenderness ratio for which instability in bending will predominate over that due to bulging in a radially symmetric mode; namely,

$$4{\cdot}39 < (L_0/R_0)_{\lim} \leq 4{\cdot}62 \text{ for cylindrical bars.} \tag{7}$$

At the lower extreme in (7), the first discernible departure from a state of uniform compression in the major central portion of the test-piece M-5 appeared as a lateral contraction near the mid-section, but further loading produced an axial curvature characteristic of buckling in bending. This phenomenon is shown in Fig. 6. Notice that the concave side eventually kinked sharply upon itself. We emphasize, however, that on the force–time record there was no evidence of the existence of a maximum collapse load.

At the other extreme, shown in Fig. 7, the specimen M*-3 exhibited bending which increased continuously with the loading until ultimately collapse occurred at a well-defined maximum pressure. Eventually, the concave side kinked sharply upon itself similar to the cylinder shown in Fig. 6.

Though (7) certainly tempts one to examine the behavior of a bar with slenderness ratio in the neighborhood of 4·5, it seemed to us pointless to do so, for additional data could be neither more nor less instructive. The region (7) plainly constitutes a sensitive transition region in which the nature of the strut behavior becomes increasingly ambiguous.

4.4 *Comparison of materials with neo-Hookean model*

It is easy to compare the observed stress–deformation behavior of the materials used in our experiments with that of the neo-Hookean model. If our materials were neo-Hookean then a graph of $3P/A_0$ vs. $(\lambda - 1/\lambda^2)$ constructed from experimental data for simple tension and compression ought to yield a straight line, in which case the elastic modulus E is determined by its slope. Upon plotting the data in this way, we found that for all materials the compression data was linear; however, except for small extensions, the tension data* was not. Nevertheless, these graphs could still be used to determine approximate values for the elastic modulus. These are given in Table 1. A typical graph is shown in Fig. 8,

* Simple tension experiments were performed on circular rods and on flat, dumb-bell shaped specimens prepared from the same vulcanizates as the compression samples. Tensile loads and extensions were determined in the same way as in the compression tests.

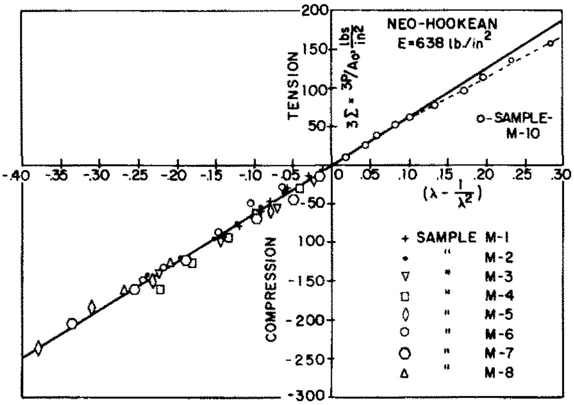


FIG. 8. Typical stress–deformation data compared with ideal neo-Hookean material.

where the compression data includes only those values of $\lambda > \lambda_{cr}$ and where only a portion of the tension curve is shown so as to allow for appropriate magnification of the diagram. We note that these curves, including the deviation from neo-Hookean behavior in tension, are similar to those described by Treloar [5, p. 94] as being typical of most rubber-like materials. Consequently, the neo-Hookean description provides a reasonable though not fully satisfactory model of our materials within the range of deformations required here.

5. COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS

In this section we compare our experimental data with theoretical analyses due to Beatty [1], Fosdick and Shield [2] and Read [3]. As noted earlier, an energy criterion was used in [1]; the others have employed Euler's criterion of existence of an adjacent equilibrium configuration, first introduced in finite elasticity by Wilkes [4].

For various end conditions, Fosdick and Shield [2] have obtained general solutions to the problem of small bending of a circular bar of incompressible, isotropic material that has been initially finitely extended or compressed by uniform axial dead loads. They investigate numerically Euler buckling for Mooney–Rivlin and neo-Hookean materials, and show that the well-known Euler formulae are in surprisingly good agreement with their results, not only for long bars that buckle at small compressive strains but also for relatively short bars. They also find that all cylinders of Mooney–Rivlin or neo-Hookean material, regardless of end conditions, are unstable in bending for $\lambda \leq 0{\cdot}444$. This corresponds roughly to a ratio $L_0/R_0 \leq 3{\cdot}0$ (cf. [3]).

Euler buckling of a neo-Hookean circular column with ends fixed so that end cross-sections remain flat and parallel to their original unstrained positions, a constraint that approximates the end conditions realized in our experiments, has also been studied by Read [3]. Though his analysis is different from that of Fosdick and Shield [2], his results are the same.

Radially symmetric buckling of a neo-Hookean strut has been examined by Wilkes [4]. In contrast with the solutions gotten by Fosdick and Shield, and Read, which satisfy overall loading conditions or approximate displacement conditions on the ends of the cylinder, Wilkes has shown that for rods long enough for end effects to be ignored, symmetric buckling will not occur unless $\lambda \leq 0{\cdot}446$. This result implies that buckling in bending will predominate over buckling in symmetric modes for all "sufficiently long" but not necessarily thin neo-Hookean bars. However, the least slenderness ratio for which Euler buckling will prevail is undetermined.

Since the theoretical critical compression ratio given by $(4)_1$, $(5)_1$ and the numerical results of Fosdick and Shield and Read are not explicitly dependent on E, a graph of the critical deformation ratio vs. the slenderness ratio provides a convenient means of comparing all experiments described in Table 1 with the aforementioned theoretical results. This graph* is shown in Fig. 9, where the computations provided in [3] have been used to construct the solution curves found by Fosdick and Shield and Read. The usual Euler solutions may be found in [11].

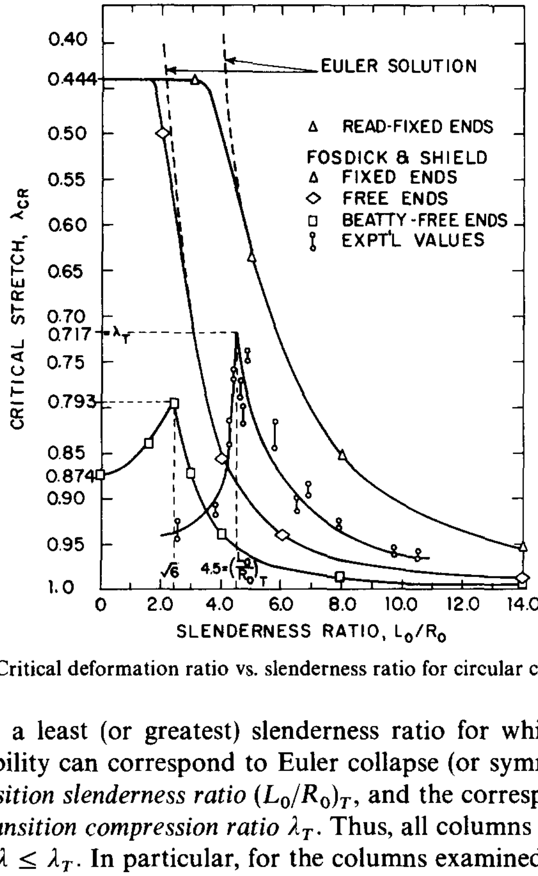


FIG. 9. Critical deformation ratio vs. slenderness ratio for circular columns.

The graph shows a least (or greatest) slenderness ratio for which the predominant initial mode of instability can correspond to Euler collapse (or symmetric buckling). We christen this the *transition slenderness ratio* $(L_0/R_0)_T$, and the corresponding compression ratio we name the *transition compression ratio* λ_T. Thus, all columns subject to end thrust are unstable for any $\lambda \leq \lambda_T$. In particular, for the columns examined here, an estimate of the *transition point* is $(L_0/R_0)_T \doteq 4{\cdot}5$, $\lambda_T \doteq 0{\cdot}717$. This point does not correspond, as estimated by Beatty [1], to a deformed geometry for which all principal values of the inertia tensor are equal. To satisfy this criterion at $(L_0/R_0)_T = 4{\cdot}5$, we ought to have $\lambda_T = 0{\cdot}529$. It seems from our experiments that such a value could not be attained; nevertheless, we certainly cannot conclude from a single rough, graphical estimate of λ_T that Beatty's characterization of the transition point is altogether wrong, only unlikely. On the other hand, as is plain from Fig. 9, the estimates found by Beatty [1] are in remarkably good

* A plot of P_{cr}/A_0E against L_0/R_0 based upon $(4)_2$ and $(5)_2$ yields curves similar to those shown in Fig. 9.

qualitative agreement with our overall observations. This, and no more, is what we proposed to show here. Thus our experiments give credence to the conclusion that the theoretical limit values $\lambda_{cr} = 0{\cdot}793$, $L_0/R_0 = 2{\cdot}45$, which divide his inequalities (1) and (2), correspond to transition ratios which separate buckling in bending from buckling in symmetric modes. Like the classical result for free end bars, his estimate of the Euler collapse load, as anticipated, is well below the experimental values found here.

We repeat: By our present experimental procedure evaluation of commencement of instability for the short, rubber cylinders with $L_0/R_0 < 4{\cdot}5$ was especially difficult; and in spite of considerable care exercised in conducting these tests and in analyzing the data, we do not know to what extent end effects have influenced the results. Although the column ends were well lubricated, some frictional constraint was always present. Consequently, it is not unlikely that this led to a premature apparent bulging effect, hence, possibly to a greater measured value for the critical compression ratio than one might find under more ideal conditions. We cannot tell from the present trials whether λ_{cr} approaches a maximum value as L_0/R_0 approaches zero, as is indicated by (2) or (5).

Though we make no claim that our data for the critical deformations is altogether precise in magnitude, we firmly believe our general observations of material behavior to be accurate. In particular, we have found that only those columns that fail of stability by bending experience an ultimate collapse load that is characterized by a well-defined maximum end-thrust. Since this information could be accurately determined from machine recorded force–time curves like that portrayed in Fig. 3, and is not dependent upon strain measurements in the critical range, it is our opinion that the graph shown in Fig. 10 lends to our observations and conclusions unequivocal support. We also note that some preliminary experiments on rectangular rods have yielded results similar to those discussed herein.

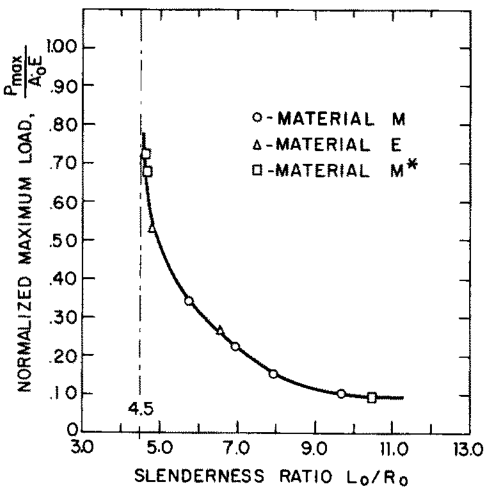


FIG. 10. Normalized maximum load vs. slenderness ratio for circular cylinders exhibiting Euler buckling.

The theoretical results found by Read [3] and Fosdick and Shield [2] do not describe the general behavior of very short struts. On the other hand, for rods that suffer Euler collapse their results, and also the classical formulae, are in good qualitative agreement with our experiments, the data falling between the values predicted for fixed end and free end cylinders. Owing to various kinds of imperfections, such as some unavoidable initial curvature of the column, non-homogeneity or anisotropy of the material, failure to satisfy ideal dead loading or other end conditions, and even the ubiquitous personal equation, we certainly do not expect precise conformity between observation and theory. However, with improved experimental techniques that insure that end conditions approach more closely theoretical assumptions and materials that correspond more nearly to the ideal model, it may be possible to reach closer coincidence with the theoretical values of Fosdick and Shield, and Read, but only for values of $L_0/R_0 > (L_0/R_0)_T$.

Our experiments indicate material response unaccounted for by Wilkes [4]. One should not conclude from his result that Euler buckling will be the dominant mode of instability for bars with slenderness ratios as small as those implied by the least value $\lambda_{cr} = 0{\cdot}444$ found by others [2, 3]; rather, we expect that an improved analysis of radially symmetric buckling will yield a solution curve similar to that found by Beatty [1] and exemplified by experiment. The point where this curve intersects that representing the critical values found from [2, 3] for Euler collapse under corresponding end conditions will determine the transition point. Precise analytical evaluation and characterization of the transition point is presently under study.

In conclusion, it is worth noting that von Kármán (cf. [11, 12]) showed by experiments that Euler's formula for free end columns can be applied in the case of short ($L/T > 45$?) rectangular steel bars in which the critical stress exceeds the proportional limit of the material; and that, more recently, Holden [13] has proved for circular rods that the usual engineering approximation is valid provided $L/R \gg \sqrt{90}$. But never before have the classical results been verified to be valid approximations for such short, highly elastic columns as we have considered here.

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Абстракт—Приводятся некоторые эксперименты, касающиеся устойчивости толстых, круглых стержней из каучука, подверженных давлению на концах. Результаты сравниваются с разными аналитическими решениями, основанными на общих теориях упругой устойчивости. Результаты указывают на наличие критического переходного отношения гибкости, которое разделяет известную форму потери устойчивости по Эйлеру от некоторых форм кругосимметрической неустойчивости. Показываются, что теоретические оценки, полученные Витти [1], являются в хорошом качественном согласии с результатами настоящей работы. Указывается, также, почти надлежащая сходимость с решениями Фосдика-Шилда [2], Реда [3] и, с хорошо известной формулой продольного изгиба Эйлера, даже для относительно коротких балок—но только для критических отношений при давлении по сравнению большем с переходным отношением гибкости. Показано, что для кругосимметрического продольного изгиба, результат Уилкеса [4] согласен с экспериментальными наблюдениями.